Hybrid High-Order Schemes on General Meshes for Elliptic PDEs

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Key ideas for HHO

- Degrees of freedom (DOFs)
  - polynomials of order $k \geq 0$ on all mesh cells and faces
  - cell DOFs can be eliminated by static condensation
  - global discrete system on mesh skeleton
- Building principles
  - discrete differential operators based on local DOFs
  - nonconforming scheme
  - face-based penalty linking cell- and face-DOFs
- Main benefits from proposed approach
  - can handle (fairly) general 3D polyhedral meshes
  - high-order method: energy-error estimate of order $(k+1)$ and potential-error estimate of order $(k+2)$ for smooth solutions
  - SPD linear system
  - compact stencil: faces neighbors, no nodal unknowns
- **References**
  - diffusion: *Comput. Methods Appl. Math.*, 2014
  - quasi-incompressible linear elasticity: `hal-00979435`

Overview: general meshes

- Low-order schemes ($k = 0$)
  - **(MFD)** Mimetic Finite Differences [Brezzi, Lipnikov & Shashkov 05]
  - **(HFV)** Hybrid Finite Volumes [Eymard, Gallouët & Herbin 10]
  - **(MFV)** Mixed Finite Volumes [Droniou & Eymard 06]
  - unified approach to MFD/HFV/MFV [Droniou et al. 10]
  - **(CDO)** Compatible Discrete Operator [Bonelle & AE 14]; vertex- and cell-based versions, hybridization, links with MFD/HFV/MFV

- Higher-order schemes ($k \geq 1$)
  - **(IPDG)** Interior Penalty Discontinuous Galerkin [Arnold et al. 01]
  - FEM w/ **nonpolynomial** shape functions [Tabarrei & Sukumar 04]
  - High-order **MFD** [Beirão da Veiga, Lipnikov & Manzini 11]
  - **(VEM)** Virtual Element Method [Brezzi, Marini et al. 12-]
  - the last three schemes aim at conformity
  - DG and HHO schemes are nonconforming

Overview: Face-based DOFs for diffusion

- **HHO** with $k = 0$ corresponds to HFV w/ specific penalty value
- Face-based DOFs for diffusion considered in
  - HDG (Hybrid DG) [Cockburn, Gopalakrishnan & Lazarov 09]
  - Generalized Cell Boundary Element method [Jeon & Park 10, 13]
  - Weak Galerkin scheme [Wang & Ye 13]
  - MFD [Lipnikov & Manzini 14], nonconforming VEM [Ayuso et al 14]
  - Hybrid-Mixed method [Araya, Harder, Paredes & Valentin 13]
- HHO differs from above in design and analysis
  - based on primal formulation
  - reconstructed gradients are locally curl-free
  - local Neumann problems on simple polynomial spaces
  - high-order (special) stabilization

Diffusion

- Model problem
- Admissible mesh sequences
- Degrees of freedom
- Local reconstructions
- Discrete problem
- Analysis: stability and convergence
- Numerical results

Model problem

- Open, bounded, connected, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$
- Source term $f \in L^2(\Omega)$
- Weak formulation: Seek $u \in H^1_0(\Omega)$ such that

$$(\nabla u, \nabla v)_\Omega = (f, v)_\Omega \qquad \forall v \in H^1_0(\Omega)$$

  u is called the **potential** and $-\nabla u$ the **flux**

- Extensions to other BCs and more general diffusion (anisotropic/heterogeneous) can be considered

Admissible mesh sequences

- h-refined mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ where each $\mathcal{T}_h$ consists of 3D polyhedral cells partitioning Ω
- Each $\mathcal{T}_h$ admits a matching simplicial submesh with only one length scale locally (cellwise)
  - submesh serves for theoretical analysis and for quadratures
  - generic constants C can depend on mesh regularity
- Usual inverse, trace, and polynomial approximation properties hold on admissible mesh sequences (see, e.g., [Di Pietro & AE 12])

Degrees of freedom

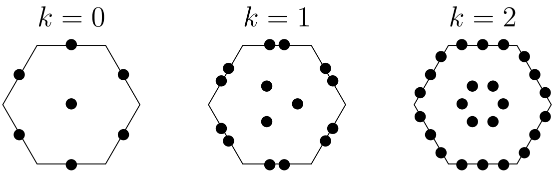
- Local DOFs are, for all $T \in \mathcal{T}_h$,

$$\mathrm{U}_T^k := \mathbb{P}_d^k(T) \times \left\{ \times_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F) \right\}$$

We use the notation $(\mathrm{v}_T, (\mathrm{v}_F)_{F \in \mathcal{F}_T})$ for $\mathrm{v} \in \mathrm{U}_T^k$

- Local reduction map $\mathrm{I}_T^k : H^1(T) \to \mathrm{U}_T^k$ such that, for all $v \in H^1(T)$,

$$\mathrm{I}_T^k v := (\pi_T^k v, (\pi_F^k v)_{F \in \mathcal{F}_T})$$



Local reconstructions (1)

- Local potential reconstruction operator $p_T^k : \mathsf{U}_T^k \to \mathbb{P}_d^{k+1}(T)$
- Let $\mathsf{v} := (\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T})$; then, $p_T^k\mathsf{v}$ solves the **local (well-posed) Neumann problem**

$$(\nabla(p_T^k\mathsf{v}), \nabla q)_T = (\nabla \mathsf{v}_T, \nabla q)_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F - \mathsf{v}_T, \nabla q{\cdot}\underline{n}_{TF})_F$$

for all $q \in \mathbb{P}_d^{k+1}(T)$, and we prescribe $\int_T (p_T^k\mathsf{v}) = \int_T \mathsf{v}_T$

- Local gradient reconstruction operator $\underline{G}_T^k : \mathsf{U}_T^k \to \nabla\mathbb{P}_d^{k+1}(T)$ s.t.

$$\underline{G}_T^k\mathsf{v} := \nabla(p_T^k\mathsf{v})$$

Local reconstructions (2)

- **Compatible discretization** (commuting diagram)

$$
\begin{array}{ccc}
H^1(T) & \xrightarrow{\ \boldsymbol{\nabla}\ } & L^2(T)^d \\
\big\downarrow \mathrm{I}_T^k & & \big\downarrow \pi_{\boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)} \\
\mathrm{U}_T^k & \xrightarrow{\ \underline{G}_T^k\ } & \boldsymbol{\nabla}\mathbb{P}_d^{k+1}(T)
\end{array}
$$

For all $u \in H^1(T)$ and all $q \in \mathbb{P}_d^{k+1}(T)$,

$$(\boldsymbol{\nabla}(p_T^k \mathrm{I}_T^k u), \boldsymbol{\nabla} q)_T = (\underline{G}_T^k \mathrm{I}_T^k u, \boldsymbol{\nabla} q)_T = (\boldsymbol{\nabla} u, \boldsymbol{\nabla} q)_T$$

- Interpolation operator $p_T^k \mathrm{I}_T^k : H^1(T) \to \mathbb{P}_d^{k+1}(T)$ with optimal approximation properties for all $k \geq 0$,

$$\|u - p_T^k \mathrm{I}_T^k u\|_T + h_T^{1/2} \|u - p_T^k \mathrm{I}_T^k u\|_{\partial T} + h_T \|\boldsymbol{\nabla}(u - p_T^k \mathrm{I}_T^k u)\|_T + h_T^{3/2} \|\boldsymbol{\nabla}(u - p_T^k \mathrm{I}_T^k u)\|_{\partial T} \leq C h_T^{k+2} \|u\|_{H^{k+2}(T)}$$

Discrete problem (1)

- Global DOFs obtained by patching interface values

$$\mathsf{U}_h^k := \left\{\times_{T\in\mathcal{T}_h}\mathbb{P}_d^k(T)\right\} \times \left\{\times_{F\in\mathcal{F}_h}\mathbb{P}_{d-1}^k(F)\right\}$$

  We use the notation $((\mathsf{v}_T)_{T\in\mathcal{T}_h}, (\mathsf{v}_F)_{F\in\mathcal{F}_h})$ for $\mathsf{v}_h \in \mathsf{U}_h^k$

- $\mathsf{L}_T : \mathsf{U}_h^k \to \mathsf{U}_T^k$ maps global to local DOFs

- Dirichlet BCs can be embedded in discrete space

$$\mathsf{U}_{h,0}^k := \left\{\mathsf{v}_h \in \mathsf{U}_h^k \mid \mathsf{v}_F \equiv 0\ \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$$

Discrete problem (2)

- Local bilinear forms on $\mathrm{U}_T^k \times \mathrm{U}_T^k$ such that

$$a_T(\mathrm{u},\mathrm{v}) := (\underline{G}_T^k\mathrm{u}, \underline{G}_T^k\mathrm{v})_T + s_T(\mathrm{u},\mathrm{v})$$

$$s_T(\mathrm{u},\mathrm{v}) := \sum_{F\in\mathcal{F}_T} h_F^{-1}(\pi_F^k(\mathrm{u}_F - \hat{p}_T^k\mathrm{u}), \pi_F^k(\mathrm{v}_F - \hat{p}_T^k\mathrm{v}))_F$$

with $\hat{p}_T^k\mathrm{v} := \mathrm{v}_T + \underbrace{(p_T^k\mathrm{v} - \pi_T^k p_T^k\mathrm{v})}_{\text{high-order correction}}$ for all $\mathrm{v} \in \mathrm{U}_T^k$

- Global bilinear form on $\mathrm{U}_h^k \times \mathrm{U}_h^k$ is **assembled cellwise**

$$a_h(\mathrm{u}_h, \mathrm{v}_h) := \sum_{T\in\mathcal{T}_h} a_T(\mathrm{L}_T\mathrm{u}_h, \mathrm{L}_T\mathrm{v}_h)$$

- Discrete problem: Find $\mathrm{u}_h \in \mathrm{U}_{h,0}^k$ such that, for all $\mathrm{v}_h \in \mathrm{U}_{h,0}^k$,

$$a_h(\mathrm{u}_h, \mathrm{v}_h) = \ell_h(\mathrm{v}_h) := \sum_{T\in\mathcal{T}_h} (f, \mathrm{v}_T)_T$$

Analysis: stability

- **Energy-norm** $\|\cdot\|_{1,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathsf{L}_T \cdot\|_{1,T}^2$ where

$$\|\mathsf{v}\|_{1,T}^2 := \|\nabla \mathsf{v}_T\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1}\|\mathsf{v}_F - \mathsf{v}_T\|_F^2 \qquad \forall \mathsf{v}\in \mathsf{U}_T^k$$

- **Stability**: There is $\eta > 0$ s.t., for all $T\in\mathcal{T}_h$,

$$\eta^{-1}\|\mathsf{v}\|_{1,T}^2 \leq a_T(\mathsf{v},\mathsf{v}) \leq \eta\|\mathsf{v}\|_{1,T}^2 \qquad \forall \mathsf{v}\in \mathsf{U}_T^k$$

- The discrete problem is well-posed

Analysis: convergence

- Energy-estimate: setting $\mathrm{I}_h^k u = ((\pi_T^k u)_{T\in\mathcal{T}_h}, (\pi_F^k u)_{F\in\mathcal{F}_h})$,

$$\|\mathrm{I}_h^k u - \mathrm{u}_h\|_{1,h} \le Ch^{k+1}\|u\|_{H^{k+2}(\Omega)}$$

  - consistency error $\mathcal{E}_h(\mathrm{v}_h) := a_h(\mathrm{I}_h^k u, \mathrm{v}_h) - \ell_h(\mathrm{v}_h)$ for all $\mathrm{v}_h \in \mathrm{U}_{h,0}^k$
  - corollary: $\|\nabla u - \underline{G}_h^k \mathrm{u}_h\|_{L^2(\Omega)} \le Ch^{k+1}\|u\|_{H^{k+2}(\Omega)}$ with discrete gradient $\underline{G}_h^k \mathrm{u}_h$ assembled cellwise

- L^2-estimate: Assuming elliptic regularity (and $f \in H^1(\Omega)$ if $k = 0$),

$$\left\{\sum_{T\in\mathcal{T}_h} \|\pi_T^k u - \mathrm{u}_T\|_T^2\right\}^{1/2} \le Ch^{k+2}\|u\|_{H^{k+2}(\Omega)}$$

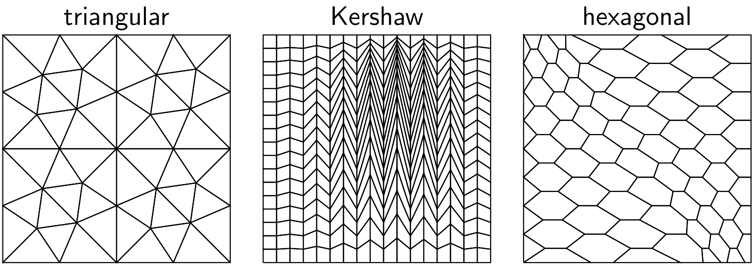
  - similar estimate as for mixed FE
  - corollary: $\|u - p_h^k \mathrm{u}_h\|_{L^2(\Omega)} \le Ch^{k+2}\|u\|_{H^{k+2}(\Omega)}$ with discrete potential $p_h^k \mathrm{u}_h$ assembled cellwise

Remarks on implementation

- Local systems solved using Cholesky factorization (Eigen v3)
  - Monomial basis in local translated/rescaled coordinates
- Global system: PETSc interface (SuperLU) [Demmel et al. 99]
  - Dirichlet BCs are enforced by means of a Lagrange multiplier
  - simplicial submesh can be exploited for quadratures
- Qualitative comparison with IPDG
  - IPDG requires pol. order $(k+1)$ to achieve the same CV order
  - HHO uses less DOFs for $k \gg 1$ ($O(k^{d-1})\times$ #(faces) vs. $O(k^d)\times$ #(cells))
  - block-stencil for IPDG is approx. twice as small, but blocks are larger

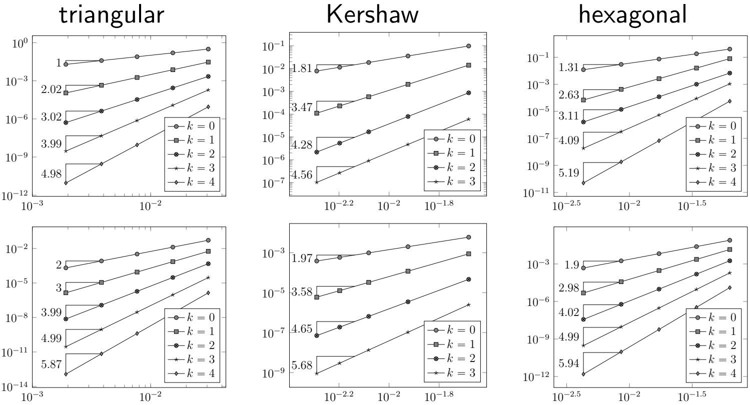
Numerical results (1)

- Dirichlet problem with smooth solution in unit square
- Mesh families from FVCA benchmark [Herbin & Hubert 08] and from [Di Pietro & Lemaire 14]



Numerical results (2)

- Energy- and L^2-norm error as a function of h



Linear elasticity

- Model problem and state of the art
- Degrees of freedom
- Local reconstructions
- Discrete problem
- Analysis: stability and convergence
- Numerical results

Model problem

- Open, bounded, connected, polyhedral domain $\Omega \subset \mathbb{R}^d$, $d \in \{2,3\}$
- Source term $\underline{f} \in L^2(\Omega)^d$, homogeneous Dirichlet BCs
- Weak formulation: Seek $\underline{u} \in H_0^1(\Omega)^d$ such that

$$(2\mu \nabla_{\mathrm{s}} \underline{u}, \nabla_{\mathrm{s}} \underline{v})_\Omega + (\lambda \nabla{\cdot}\underline{u}, \nabla{\cdot}\underline{v})_\Omega = (\underline{f}, \underline{v})_\Omega \qquad \forall \underline{v} \in H_0^1(\Omega)^d$$

  with scalar Lamé coefficients $\mu > 0$ and $\lambda \geq 0$ and ∇_{s} denoting the symmetric part of gradient operator
- $\underline{u}$ is the **displacement** field, $\underline{\underline{\varepsilon}} = \nabla_{\mathrm{s}} \underline{u}$ the (linearized) **strain** tensor, and $\underline{\underline{\sigma}} = 2\mu \nabla_{\mathrm{s}} \underline{u} + \lambda (\nabla{\cdot}\underline{u}) \underline{\underline{I}}_d$ the **stress** tensor

Quasi-incompressible limit

- Quasi-incompressible limit $\lambda \to +\infty$ requires discrete space to accurately represent nontrivial divergence-free fields
  - locking phenomenon for classical conforming FE
- Nonconforming primal methods on **specific** meshes
  - CR [Brenner & Sung 92], IPDG [Hansbo & Larson 02-03]
  - HDG with strongly symmetric stresses [Qiu & Shi 14]
- Low-order methods on general meshes
  - MFD [Beirão da Veiga, Gyrya, Lipnikov & Manzini 09]
  - generalized CR [Di Pietro & Lemaire 14]
  - approximate gradient schemes [Droniou & Lamichhane 14]
- VEM on general meshes for planar elasticity with vertex-, edge-, and cell-based DOFs [Beirão da Veiga, Brezzi & Marini 13]
- HHO with $k \geq 1$ on general 3D meshes

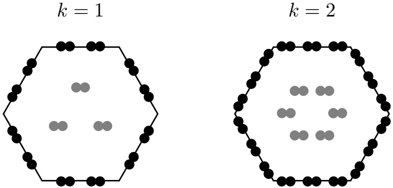
Degrees of freedom

- Admissible mesh sequence; local DOFs are, for all $T \in \mathcal{T}_h$,

$$\underline{\mathsf{U}}_T^k := \mathbb{P}_d^k(T)^d \times \left\{ \times_{F \in \mathcal{F}_T} \mathbb{P}_{d-1}^k(F)^d \right\}$$

- Local reduction map $\mathsf{I}_T^k : H^1(T)^d \to \underline{\mathsf{U}}_T^k$ such that

$$\mathsf{I}_T^k \underline{v} = (\pi_T^k \underline{v}, (\pi_F^k \underline{v})_{F \in \mathcal{F}_T})$$



Local reconstructions (1)

- Local displacement reconstruction operator $p_T^k : \underline{U}_T^k \to \mathbb{P}_d^{k+1}(T)^d$
- Let $\underline{v} := (\underline{v}_T, (\underline{v}_F)_{F\in\mathcal{F}_T})$; then, $p_T^k\underline{v}$ solves the **local (well-posed) Neumann problem**

$$(\nabla_{\mathrm{s}}(p_T^k\underline{v}), \nabla_{\mathrm{s}}\underline{q})_T = (\nabla_{\mathrm{s}}\underline{v}_T, \nabla_{\mathrm{s}}\underline{q})_T + \sum_{F\in\mathcal{F}_T} (\underline{v}_F - \underline{v}_T, \nabla_{\mathrm{s}}\underline{q}\,\underline{n}_{TF})_F$$

for all $\underline{q} \in \mathbb{P}_d^{k+1}(T)^d$, with rigid-body motions of $p_T^k\underline{v}$ prescribed from $\underline{v}$

- Local symmetric gradient reconstruction $\underline{\underline{E}}_T^k : \underline{U}_T^k \to \nabla_{\mathrm{s}}\mathbb{P}_d^{k+1}(T)^d$ s.t.

$$\underline{\underline{E}}_T^k\underline{v} := \nabla_{\mathrm{s}}(p_T^k\underline{v})$$

Local reconstructions (2)

- **Compatible discretization** (commuting diagram)

$$
\begin{array}{ccc}
H^1(T)^d & \xrightarrow{\ \boldsymbol{\nabla}_{\mathrm{s}}\ } & L^2(T)^{d\times d} \\
\Big\downarrow \mathsf{I}_T^k & & \Big\downarrow \pi_{\boldsymbol{\nabla}_{\mathrm{s}}\mathbb{P}_d^{k+1}(T)^d} \\
\underline{\mathsf{U}}_T^k & \xrightarrow{\ \underline{\underline{E}}_T^k\ } & \boldsymbol{\nabla}_{\mathrm{s}}\mathbb{P}_d^{k+1}(T)^d
\end{array}
$$

For all $\underline{u} \in H^1(T)^d$ and all $\underline{q} \in \mathbb{P}_d^{k+1}(T)^d$,

$$
(\boldsymbol{\nabla}_{\mathrm{s}}(p_T^k \mathsf{I}_T^k \underline{u}), \boldsymbol{\nabla}_{\mathrm{s}}\underline{q})_T = (\underline{\underline{E}}_T^k \mathsf{I}_T^k \underline{u}, \boldsymbol{\nabla}_{\mathrm{s}}\underline{q})_T = (\boldsymbol{\nabla}_{\mathrm{s}}\underline{u}, \boldsymbol{\nabla}_{\mathrm{s}}\underline{q})_T
$$

- Interpolation operator $p_T^k \mathsf{I}_T^k : H^1(T)^d \to \mathbb{P}_d^{k+1}(T)^d$ with optimal approximation properties

$$
\|\underline{u} - p_T^k \mathsf{I}_T^k \underline{u}\|_T + h_T^{1/2}\|\underline{u} - p_T^k \mathsf{I}_T^k \underline{u}\|_{\partial T} + h_T\|\boldsymbol{\nabla}_{\mathrm{s}}(\underline{u} - p_T^k \mathsf{I}_T^k \underline{u})\|_T + h_T^{3/2}\|\boldsymbol{\nabla}_{\mathrm{s}}(\underline{u} - p_T^k \mathsf{I}_T^k \underline{u})\|_{\partial T} \le C h_T^{k+2}\|\underline{u}\|_{H^{k+2}(T)}
$$

Local reconstructions (3)

- Local divergence reconstruction operator $D_T^k : \underline{\mathsf{U}}_T^k \to \mathbb{P}_d^k(T)$
- For all $\underline{\mathsf{v}} = (\mathsf{v}_T, (\mathsf{v}_F)_{F\in\mathcal{F}_T}) \in \underline{\mathsf{U}}_T^k$, $D_T^k\underline{\mathsf{v}}$ is determined from

$$(D_T^k\underline{\mathsf{v}}, q)_T := (\nabla\cdot\mathsf{v}_T, q)_T + \sum_{F\in\mathcal{F}_T} (\mathsf{v}_F - \mathsf{v}_T, q\underline{n}_{TF})_F$$

for all $q \in \mathbb{P}_d^k(T)$

- Commuting diagram property (key for incompressible limit)

$$\begin{array}{ccc} H^1(T)^d & \xrightarrow{\nabla\cdot} & L^2(T) \\ \downarrow \mathsf{I}_T^k & & \downarrow \pi_T^k \\ \underline{\mathsf{U}}_T^k & \xrightarrow{D_T^k} & \mathbb{P}_d^k(T) \end{array}$$

Discrete problem

- Local bilinear forms on $\underline{U}_T^k \times \underline{U}_T^k$ such that

$$a_T(\underline{u}, \underline{v}) := 2\mu(\underline{\underline{E}}_T^k \underline{u}, \underline{\underline{E}}_T^k \underline{v})_T + \lambda(D_T^k \underline{u}, D_T^k \underline{v})_T + 2\mu s_T(\underline{u}, \underline{v})$$

$$s_T(\underline{u}, \underline{v}) := \sum_{F \in \mathcal{F}_T} h_F^{-1}(\pi_F^k(\underline{u}_F - \hat{p}_T^k \underline{u}), \pi_F^k(\underline{v}_F - \hat{p}_T^k \underline{v}))_F$$

with $\hat{p}_T^k \underline{v} := \underline{v}_T + (p_T^k \underline{v} - \pi_T^k p_T^k \underline{v})$ for all $\underline{v} \in \underline{U}_T^k$

- Global bilinear form a_h on $\underline{U}_h^k \times \underline{U}_h^k$ is **assembled cellwise**
- Global DOFs obtained by patching interface values, Dirichlet BCs can be embedded in discrete space

$$\underline{U}_{h,0}^k := \left\{\underline{v}_h \in \underline{U}_h^k \mid \underline{v}_F \equiv \underline{0} \; \forall F \in \mathcal{F}_h^{\mathrm{b}}\right\}$$

- Discrete problem: Find $\underline{u}_h \in \underline{U}_{h,0}^k$ such that, for all $\underline{v}_h \in \underline{U}_{h,0}^k$,

$$a_h(\underline{u}_h, \underline{v}_h) = \ell_h(\underline{v}_h) := \sum_{T \in \mathcal{T}_h} (\underline{f}, \underline{v}_T)_T$$

Analysis: stability

- Discrete strain norm $\|\cdot\|_{\varepsilon,h}^2 := \sum_{T\in\mathcal{T}_h} \|\mathsf{L}_T \cdot\|_{\varepsilon,T}^2$ where

$$\|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 := \|\boldsymbol{\nabla}_{\mathrm{s}} \underline{\mathsf{v}}_T\|_T^2 + \sum_{F\in\mathcal{F}_T} h_F^{-1} \|\underline{\mathsf{v}}_F - \underline{\mathsf{v}}_T\|_F^2 \qquad \forall \underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$$

- Stability: Let $k \geq 1$. There is $\eta > 0$ s.t., for all $T \in \mathcal{T}_h$,

$$\eta \|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 \leq \|\underline{\underline{E}}_T^k \underline{\mathsf{v}}\|_T^2 + s_T(\underline{\mathsf{v}},\underline{\mathsf{v}}) \leq \eta^{-1} \|\underline{\mathsf{v}}\|_{\varepsilon,T}^2 \qquad \forall \underline{\mathsf{v}} \in \underline{\mathsf{U}}_T^k$$

- The discrete problem is **well-posed**

Analysis: convergence

- Define energy norm as $\|\underline{v}_h\|^2_{\mathrm{en},h} := a_h(\underline{v}_h, \underline{v}_h)$, i.e.,

$$\|\underline{v}_h\|^2_{\mathrm{en},h} = \sum_{T\in\mathcal{T}_h} \{2\mu\|\underline{\underline{E}}^k_T \mathsf{L}_T \underline{v}_h\|^2_T + \lambda\|D^k_T \mathsf{L}_T \underline{v}_h\|^2_T + s_T(\mathsf{L}_T\underline{v}_h, \mathsf{L}_T\underline{v}_h)\}$$

- Energy-norm error estimate: setting $\mathsf{I}^k_h \underline{u} = ((\pi^k_T \underline{u})_{T\in\mathcal{T}_h}, (\pi^k_F \underline{u})_{F\in\mathcal{F}_h})$,

$$(2\mu)^{1/2}\|\mathsf{I}^k_h\underline{u} - \underline{\mathsf{u}}_h\|_{\mathrm{en},h} \le Ch^{k+1}\left(2\mu\|\underline{u}\|_{H^{k+2}(\Omega)} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\Omega)}\right)$$

  - C independent of h, μ, λ
  - corollary: same bound on $(2\mu)\|\nabla_{\mathrm{s}}\underline{u} - \underline{\underline{E}}^k_h \underline{\mathsf{u}}_h\|_{L^2(\Omega)}$

- L^2-norm error estimate: Assuming elliptic regularity,

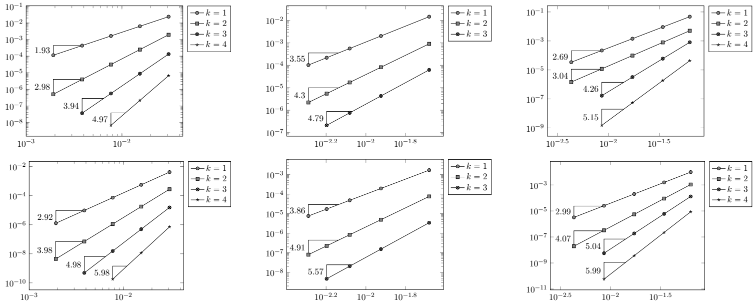
$$\left\{\sum_{T\in\mathcal{T}_h}\|\pi^k_T\underline{u} - \underline{\mathsf{u}}_T\|^2_T\right\}^{1/2} \le C_\mu h^{k+2}\left(2\mu\|\underline{u}\|_{H^{k+2}(\Omega)} + \lambda\|\nabla\cdot\underline{u}\|_{H^{k+1}(\Omega)}\right)$$

  - C_μ independent of h, λ
  - corollary: same bound on $\|\underline{u} - p^k_h\underline{\mathsf{u}}_h\|_{L^2(\Omega)}$

Numerical results (1)

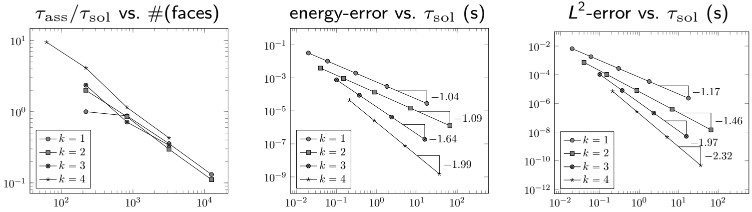
- Two-dimensional, pure-displacement problem on unit square with $\mu = 1$, $\lambda \in \{1, 1000\}$, and smooth solution
- Energy- and L^2-norm error as a function of h ($\lambda = 1000$)

triangular   Kershaw   hexagonal



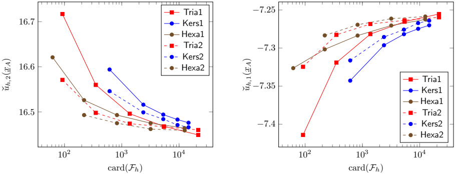
Numerical results (2)

- Performance assessment: assembly time τ_{ass}, solution time τ_{sol}
- Results for hexagonal mesh family

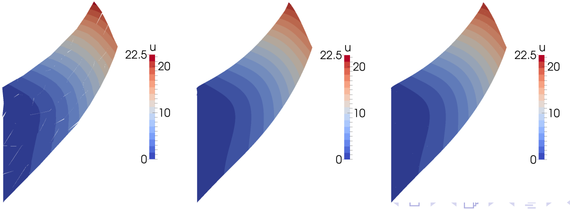


Numerical results (3)

- Cook's membrane test case ($\mu = 0.375$ and $\lambda = 7.5\times 10^6$)
- Convergence history (3 meshes, $k = 1$ and 2) for vertical and horizontal displacement of reference point



- Deformed configuration for coarsest (22 cells), intermediate (280 cells), and finest (4,192) hexagonal meshes



Conclusions and outlook

- HHO methods for linear elasticity offer several advantages
  - locking-free primal formulation, global SPD system, strongly symmetric strain and stress tensors
  - compact-stencil (face neighbors), face-based DOFs simplify data exchange (wrt to vertex-based DOFs)
  - high-order, general 3D meshes
- **Price to be paid?**
  - local problems (relative cost swiftly decreases with mesh refinement)
  - nonconforming method (post-process solution, jumps optimally converge to zero)
- In 3D, lowest-order version requires 9 DOFs per mesh face ($k = 1$)