

Méthode d'équation intégrale accélérée par la méthode multipôle rapide pour les problèmes tridimensionnels de propagation d'ondes élastiques dans un demi-espace

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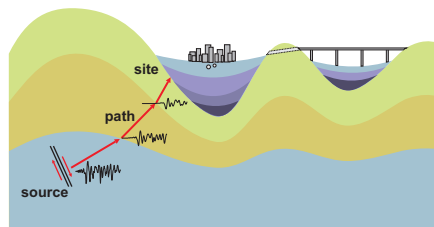
Séminaire LaMSID
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Travail effectué en collaboration avec M. Bonnet



- 1 Motivations and background
- 2 Formulation of the Elastic Half-space Fundamental solutions
- 3 New Fast Multipole Formulation for the Elastic Half-space
- 4 Numerical experiments and complexity study
- 5 Conclusions and future work

Modelling of elastic wave propagation in large/unbounded domains



- Site effects
- Soil-structure interaction
- Computational forward solution method for inverse problems

Pros and cons of Boundary Element Methods for seismic waves

Domain methods (FEM, SEM, ...)

- Domain mesh
- Approx. radiation conditions
- Sparse matrix

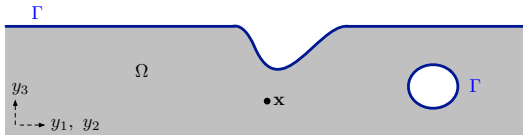
BEM

- Surface mesh (i.e. reduced dim.)
- Exact radiation conditions
- Fully-populated matrix

- BEM adequate for large (unbounded) media, simple (linear) prop
- Fully-populated BEM influence matrix: severe limiting factor

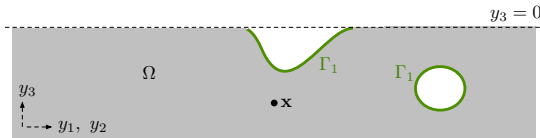
Formulation with the **full-space** fundamental solutions (F.S.)

$$c_{ik}(\mathbf{x})u_j(\mathbf{x}) = \int_{\Gamma} [t_j(\mathbf{y})U_i^k(\mathbf{x}, \mathbf{y}; \omega) - u_j(\mathbf{y})T_i^k(\mathbf{x}, \mathbf{y}; \omega)] dS_{\mathbf{y}}, \quad (\mathbf{x} \in \partial\Omega)$$



Formulation with the **half-space** fundamental solutions (F.S.)

$$c_{ik}(\mathbf{x})u_j(\mathbf{x}) = \int_{\Gamma_1} [t_i(\mathbf{y})U_i^{HS,k}(\mathbf{x}, \mathbf{y}; \omega) - u_j(\mathbf{y})T_i^{HS,k}(\mathbf{x}, \mathbf{y}; \omega)] dS_{\mathbf{y}}, \quad (\mathbf{x} \in \partial\Omega)$$



U^{HS} and T^{HS} satisfy traction-free condition on free surface $y_3 = 0$

Numerical advantages and issues

- + No truncation issue (Γ_1 is a bounded surface)
- + # DOFs **drastically reduced** compared to full-space F.S.
- **Closed-form** of U^{HS} and T^{HS} not available
- Substantial **CPU time** (integrals of oscillating functions over infinite intervals)

Fast Multipole Method (FMM)

- **Accelerates** standard BEM and **reduces** memory requirements
- Fast, approximate method for evaluating the linear integral operator (matrix-vector product, called by iterative solver)

Previous results in elastodynamics for **full-space** F.S.

- **Multi-level fast multipole method** successfully extended to single-region 3-D frequency-domain elastodynamics¹ and viscoelastodynamics²
- **Multi-region** FMM implemented for 3-D frequency-domain viscoelastodynamics (based on single-region FMM)³

¹[Chaillat *et al.*, CMAME, 2008]

²[Grasso *et al.*, EABE, 2012]

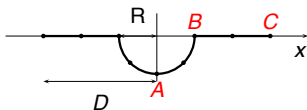
³[Chaillat *et al.*, GJI, 2009]

Numerical results in elastodynamics for **full-space** F.S.

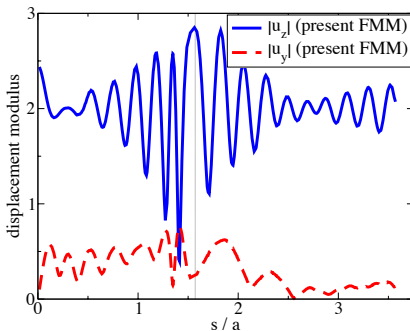
- Implementation (code COFFEE) **consistent** with theoretical complexity $O(N \log N)$, BE models of size $N = O(10^6)$
- Implementation tested against published results for low frequencies, and run for **higher** frequencies

Diffraction plane P wave by semi-spherical canyon $k_P a / \pi = 5$

$N = 287\,946$ (86 iter., 162 s per iter., single-proc. 3 GHz PC)



s/a : normalized arc-length
coordinate along ABC



Approach to define a FMM for full-space elastodynamics F.S.

- Full-space elastodynamics F.S. are derivatives of Helmholtz F.S.

$$U_i^k(\mathbf{x}, \mathbf{y}; \omega) = \frac{1}{k_{S\mu}^2} ((\delta_{qs}\delta_{ik} - \delta_{qk}\delta_{is}) \frac{\partial}{\partial x_q} \frac{\partial}{\partial y_s} G_S(|\mathbf{x} - \mathbf{y}|) + \frac{\partial}{\partial x_i} \frac{\partial}{\partial y_k} G_P(|\mathbf{x} - \mathbf{y}|))$$

$$T_i^k(\mathbf{x}, \mathbf{y}; \omega) = C_{ijhe} \frac{\partial}{\partial y_\ell} U_h^k(\mathbf{x}, \mathbf{y}; \omega) n_j(\mathbf{y})$$

$$G_\alpha(z) = \frac{\exp(ik_\alpha z)}{4\pi z} \text{ (fund. sol. Helmholtz eqn., } \alpha = P, S)$$

- Extension of FMM for Helmholtz to elastodynamics

Differences to define a FMM for half-space elastodynamics F.S.

- **No multipole expansion** of the F.S. currently available
- Need to derive half-space F.S. in form that enables **fast computat.**
- Need to formulate a **new FMM** for the elastic half-space F.S.

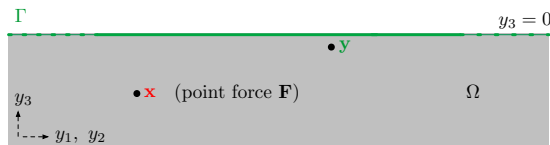
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Decomposition of the F.S. into three parts

$$\mathbf{u}^{HS}(\mathbf{x}, \mathbf{y}) = \mathbf{U}_\infty(\mathbf{x}, \mathbf{y}) + \bar{\mathbf{U}}_\infty(\mathbf{x}, \mathbf{y}) + \mathbf{U}_C(\mathbf{x}, \mathbf{y})$$

- \mathbf{U}_∞ : **elastic full-space** F.S.
- $\bar{\mathbf{U}}_\infty$: **image** elastic full-space F.S. (image point force applied at the mirror image source point)

• $\tilde{\mathbf{x}}$ (image point force $\mathbf{S} \mathbf{F}$)



- \mathbf{U}_C : **complementary** F.S.

\Rightarrow Contrary to the scalar case, the method of images allows to only partially satisfy the traction-free condition on the free-surface

Induced decomposition of integral operators (e.g. single-layer operator):

$$\mathbf{v}(\mathbf{x}) = \int_{\Gamma_1} \left(\mathbf{U}^{HS}(\mathbf{x}, \mathbf{y}) \right)^T \mathbf{t}(\mathbf{y}) dS_y, \quad \mathbf{v}(\mathbf{x}) = \mathbf{v}_\infty(\mathbf{x}) + \bar{\mathbf{v}}_\infty(\mathbf{x}) + \mathbf{v}_C(\mathbf{x})$$

Fast computation of the three contributions to $\mathbf{v}(\mathbf{x})$

- Evaluation of $\mathbf{v}_\infty(\mathbf{x})$ and $\bar{\mathbf{v}}_\infty(\mathbf{x})$: “standard” FMM (diagonal form-based decomposition of full-space F.S.⁴)
- Remaining difficulty: compute the contribution $\mathbf{v}_C(\mathbf{x})$ involving the complementary F.S.

Goals

- Formulate part \mathbf{U}_C of F.S. in a form which enables fast computat.
- Define a fast algorithm to compute the contribution $\mathbf{v}_C(\mathbf{x})$

⁴ [Chaillat *et al.*, CMAME, 2008]

General idea:

- Fourier transform against first 2 coordinates $(r_1, r_2, y_3) \rightarrow (\xi_1, \xi_2, y_3)$
- Classical result for Helmholtz equation⁵, for $y_3 > 0$

$$G(r) = \frac{\exp(ikr)}{4\pi r} = \frac{1}{2\pi} \int_0^\infty \int_0^{2\pi} e^{-y_3 \sqrt{\xi^2 - k^2}} e^{i\xi(r_1 \cos \alpha + r_2 \sin \alpha)} \frac{\xi}{\sqrt{\xi^2 - k^2}} d\xi d\alpha$$

Methodology:



- 1 Full-space F.S.: Derive stress vector $\hat{\mathbf{p}}_\infty(y_3 = 0)$
- 2 Image full-space F.S.: Derive stress vector $\hat{\hat{\mathbf{p}}}_\infty(y_3 = 0)$
- 3 Complementary term: Boundary condition

$$\hat{\mathbf{p}}_\infty + \hat{\hat{\mathbf{p}}}_\infty + \hat{\mathbf{p}}_C = 0 \quad (y_3 = 0)$$

- 4 Obtain F.S. $\hat{\mathbf{U}}_C$ in Fourier space

⁵ [Greengard *et al.*, IEEE Comp. Science. Engrg, 1998]

Expression in Fourier space

$$\hat{U}_c(\xi_1, \xi_2, y_3, x_3) = \frac{s_p}{4\pi^2 \delta(\xi) \mu k_s^2} (2\mathbf{V}_S(\xi_1, \xi_2) e^{s_s y_3} + (k_s^2 - 2\xi^2) \mathbf{V}_P(\xi_1, \xi_2) e^{s_p y_3})^T \\ \times (2\mathbf{V}'_S(\xi_1, \xi_2) e^{s_s x_3} + (k_s^2 - 2\xi^2) \mathbf{V}'_P(\xi_1, \xi_2) e^{s_p x_3})$$

$$\mathbf{V}_S = \{i\xi_1 s_s \quad i\xi_2 s_s \quad \xi^2\}, \quad \mathbf{V}_P = \{i\xi_1/s_p \quad i\xi_2/s_p \quad 1\}$$

$$s_\alpha = \sqrt{\xi^2 - k_\alpha^2}, \quad \xi^2 = \xi_1^2 + \xi_2^2$$

Expression in Fourier space

$$\hat{U}_c(\xi_1, \xi_2, y_3, x_3) = \frac{s_p}{4\pi^2 \delta(\xi) \mu k_s^2} (2\mathbf{V}_S(\xi_1, \xi_2) e^{s_s y_3} + (k_s^2 - 2\xi^2) \mathbf{V}_P(\xi_1, \xi_2) e^{s_p y_3})^T \\ \times (2\mathbf{V}'_S(\xi_1, \xi_2) e^{s_s x_3} + (k_s^2 - 2\xi^2) \mathbf{V}'_P(\xi_1, \xi_2) e^{s_p x_3})$$

$$\mathbf{V}'_S = \{i\xi_1 s_s \quad i\xi_2 s_s \quad -\xi^2\}, \quad \mathbf{V}'_P = \{i\xi_1/s_p \quad i\xi_2/s_p \quad -1\}$$

Expression in Fourier space

$$\hat{U}_C(\xi_1, \xi_2, y_3, x_3) = \frac{s_p}{4\pi^2 \delta(\xi) \mu k_S^2} (2\mathbf{V}_S(\xi_1, \xi_2) e^{s_s y_3} + (k_S^2 - 2\xi^2) \mathbf{V}_P(\xi_1, \xi_2) e^{s_p y_3})^T \\ \times (2\mathbf{V}'_S(\xi_1, \xi_2) e^{s_s x_3} + (k_S^2 - 2\xi^2) \mathbf{V}'_P(\xi_1, \xi_2) e^{s_p x_3})$$

$$\delta(\xi) = (k_S^2 - 2\xi^2)^2 - 4\xi^2 s_s s_p, \quad s_\alpha = \sqrt{\xi^2 - k_\alpha^2}$$

Expression in Fourier space

$$\hat{U}_c(\xi_1, \xi_2, y_3, x_3) = \frac{s_p}{4\pi^2 \delta(\xi) \mu k_s^2} \left(2\mathbf{V}_S(\xi_1, \xi_2) e^{s_s y_3} + (k_s^2 - 2\xi^2) \mathbf{V}_P(\xi_1, \xi_2) e^{s_p y_3} \right)^T \\ \times \left(2\mathbf{V}'_S(\xi_1, \xi_2) e^{s_s x_3} + (k_s^2 - 2\xi^2) \mathbf{V}'_P(\xi_1, \xi_2) e^{s_p x_3} \right)$$

k_R such that $\delta(k_R) = 0$ is the **Rayleigh** wavenumber ($k_P < k_S < k_R$)

Expression in Fourier space

$$\hat{U}_C(\xi_1, \xi_2, y_3, x_3) = \frac{S_P}{4\pi^2 \delta(\xi) \mu k_S^2} \left(2\mathbf{V}_S(\xi_1, \xi_2) e^{S_S y_3} + (k_S^2 - 2\xi^2) \mathbf{V}_P(\xi_1, \xi_2) e^{S_P y_3} \right)^T \\ \times \left(2\mathbf{V}'_S(\xi_1, \xi_2) e^{S_S x_3} + (k_S^2 - 2\xi^2) \mathbf{V}'_P(\xi_1, \xi_2) e^{S_P x_3} \right)$$

k_R such that $\delta(k_R) = 0$ is the **Rayleigh** wavenumber ($k_P < k_S < k_R$)

Expression in physical coordinates

$$U_C(\mathbf{x}, \mathbf{y}) = \int_0^{+\infty} \int_0^{2\pi} e^{i\xi(\cos \alpha(y_1 - x_1) + \sin \alpha(y_2 - x_2))} \hat{U}_C(\xi \cos \alpha, \xi \sin \alpha, y_3, x_3) \xi \, d\alpha \, d\xi$$

FMM-friendly formulation: integrand involves **product** of a function of \mathbf{x} and a function of \mathbf{y} → essential feature for fast evaluation

$$U_C(\mathbf{x}, \mathbf{y}) = \int_0^{+\infty} \underbrace{\int_0^{2\pi} e^{i\xi(\cos \alpha(y_1 - x_1) + \sin \alpha(y_2 - x_2))} \hat{U}_C(\xi \cos \alpha, \xi \sin \alpha, y_3, x_3) \xi \, d\alpha \, d\xi}_{I_C(\xi, \mathbf{x}, \mathbf{y})}$$

Evaluation in standard BEM (not adapted to FMM)

- Integration over α **analytical** (yields $J_i(\xi|\mathbf{y} - \mathbf{x}|)$, $0 \leq i \leq 2$)
- Destroys separation of variables \mathbf{x} and \mathbf{y}

Numerical integration in context of FMM: double integral in (α, ξ) space

- Quadrature rule over α (inner integral)
 - Smooth, periodic functions
 - ⇒ Simple **trapezoidal rule**
- Quadrature rule over ξ (integrand: $I_C(\xi, \mathbf{x}, \mathbf{y})$)
 - **Oscillatory** functions: difficult to integrate numerically
 - Singularity due to **Rayleigh pole**
 - Quadrature for \mathbf{x} and \mathbf{y} lying in a **prescribed parameter space**
 - ⇒ Non-classical quadrature rule necessary

Generalized Gaussian quadrature rules

- ⇒ Well adapted to the definition of the FMM
 - Definition of a FMM at low frequencies [Greengard *et al.*, 1998]
 - Definition of a wideband FMM [Cheng *et al.*, 2006]
- ⇒ Construction of quadrature rule is involved and integrand-specific

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Generic methodology for constructing GGQ rule [Bremer *et al.*, 2010]

- Output quadrature formula with N_ξ nodes

$$\sum_{j=1}^{N_\xi} \mathbf{I}_C(\xi_j, \mathbf{x}, \mathbf{y}) w_j \simeq \int_0^{\xi_{max}} \mathbf{I}_C(\xi, \mathbf{x}, \mathbf{y}) d\xi$$

- Quadrature integrating collection of **input functions** ϕ_1, \dots, ϕ_m
 - m square integrable functions
 - Reproduce the **singular** and **oscillatory** behaviors of \mathbf{I}_C on $[0, \xi_{max}]$
 - Sampled values of \mathbf{x} and \mathbf{y} in the **prescribed parameter space**
- ⇒ Large number m of input functions for a good accuracy
- N_ξ is **small** if the **rank** of the input functions is small

⇒ Developed a version of GGQ **custom-tailored** to our needs

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Diagonal form-based decomposition of the complementary term

$$\mathbf{U}_C(\mathbf{x}, \mathbf{y}) = \frac{1}{4\pi^2 \mu k_S^2} \sum_{a,b=P,S} \int_0^{\xi_{max}} \int_0^{2\pi} e^{i\xi(y_1 \cos \alpha + y_2 \sin \alpha)} e^{y_3 s_a} \mathcal{O}_{ab}(\xi) \mathbf{V}'_b(\xi, \alpha) \mathbf{V}_a^T(\xi, \alpha) e^{i\xi(-x_1 \cos \alpha - x_2 \sin \alpha)} e^{x_3 s_b} d\alpha d\xi$$

where $\mathcal{O}_{SS}(\xi) = \frac{4\xi s_P}{\delta}$, $\mathcal{O}_{SP}(\xi) = \mathcal{O}_{PS}(\xi) = \frac{2\xi \beta s_P}{\delta}$, $\mathcal{O}_{PP}(\xi) = \frac{\xi \beta^2 s_P}{\delta}$.

Advantages of the Multipole Expansion

- FMM-friendly formulation: **separation** of \mathbf{x} and \mathbf{y}
- **Rank one** formulation

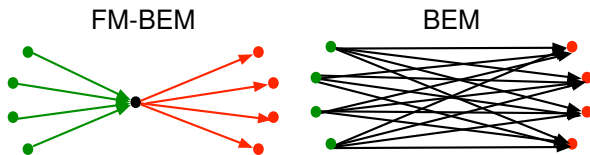
Evaluation of single-layer potential: $\mathbf{v}_C(\mathbf{x}) = \int_{\Gamma_1} \mathbf{U}_C^T(\mathbf{x}, \mathbf{y}) \mathbf{t}(\mathbf{y}) dS_y$

- 1 Computation of multipole moments ($a, b = P, S$)

$$\mathcal{R}_{ab}(\xi, \alpha) = \int_{\Gamma_1} e^{i\xi(y_1 \cos \alpha + y_2 \sin \alpha)} e^{y_3 s_a} \mathcal{O}_{ab}(\xi) \mathbf{V}'_b(\xi, \alpha) \mathbf{t}(\mathbf{y}) dS_y$$

- 2 Evaluation at observation points (quadrature in Fourier space)

$$\mathbf{v}_C^k(\mathbf{x}) = \sum_{\ell=1}^{N_\xi} \sum_{j=1}^{N_\alpha} w_j^\alpha w_\ell^\xi e^{i\xi_\ell(-x_1 \cos \alpha_j - x_2 \sin \alpha_j)} \sum_{a,b=P,S} \mathbf{V}_a^k(\xi_\ell, \alpha_j) e^{x_3 s_b} \mathcal{R}_{ab}(\xi_\ell, \alpha_j)$$



Current assumption: Γ_1 does not intercept the free-surface

- Multipole Expansion for \mathbf{U}_C always valid
- Otherwise: Singularities if $\mathbf{x} = \mathbf{y}$ on the free-surface (requires additional work)

$$\int_0^{\xi_{max}} \int_0^{2\pi} e^{i\xi(y_1 \cos \alpha + y_2 \sin \alpha)} e^{y_3 s_a} \mathcal{O}_{ab}(\xi) \mathbf{V}'_b(\xi, \alpha) \mathbf{V}_a^T(\xi, \alpha) e^{i\xi(-x_1 \cos \alpha - x_2 \sin \alpha)} e^{x_3 s_b} d\alpha d\xi$$



Complexity Estimate: computation of \mathbf{U}_C

- N : number of BE DOFs
- N_α, N_ξ : number of quadrature points inner and outer integrals
- For each quadrature point and each DOF
- Evaluating \mathbf{v}_C costs $O(N_\xi N_\alpha N)$

\Rightarrow Numerical tests to determine complexities of N_α and N_ξ with N

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Verification of the new multipole expansion in terms of

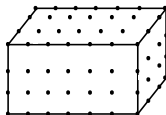
- Accuracy
- Complexity of the **matrix-vector product**

$$\sum_j^N \mathbf{U}_C(\mathbf{x}_i, \mathbf{y}_j) \mathbf{t}(\mathbf{y}_j) \quad (\text{for given sets of points } \mathbf{x}_i, \mathbf{y}_j \quad 1 \leq i, j \leq N)$$

Discretization: Fixed density of points per S-wavelength

⇒ The circular frequency ω is increased to increase # of DOFs

Rectangular domain: (non-dimensional spatial coordinates)



$$-4 \leq y_1, y_2 \leq 4, \quad 1 \leq y_3 \leq 4$$

Mechanical properties: $\mu = 3, \quad \nu = 0.25, \quad \rho = 1$

$$c_S = \sqrt{\frac{\mu}{\rho}}, \quad c_P = c_S \sqrt{\frac{2-2\nu}{1-2\nu}}, \quad k_P = \frac{\omega}{c_P}, \quad k_S = \frac{\omega}{c_S}$$

Comparisons with the code provided by B.B. Guzina⁶ (F.S. of the layered elastic half-space)

$$\varepsilon_{ij} = \frac{\| \mathbf{U}_{ij}^{HS} - (\mathbf{U}_{ij}^{HS})_{guz.} \|_2}{\| (\mathbf{U}_{ij}^{HS})_{guz.} \|_2} \quad (\text{if inner integral very accurate})$$

Accuracy w.r.t N_ξ (fixed $\omega = 1$)

N_ξ	ε_{11}	ε_{12}	ε_{13}	ε_{31}	ε_{33}
12	$4.1 \cdot 10^{-2}$	$1.9 \cdot 10^{-2}$	$1.5 \cdot 10^{-1}$	$2.1 \cdot 10^{-1}$	$2.7 \cdot 10^{-2}$
15	$4.4 \cdot 10^{-3}$	$3.1 \cdot 10^{-3}$	$2.3 \cdot 10^{-2}$	$4.8 \cdot 10^{-2}$	$1.8 \cdot 10^{-2}$
16	$1.9 \cdot 10^{-3}$	$6.3 \cdot 10^{-4}$	$4.8 \cdot 10^{-3}$	$6.3 \cdot 10^{-3}$	$1.5 \cdot 10^{-3}$
17	$1.2 \cdot 10^{-3}$	$4.2 \cdot 10^{-4}$	$3.2 \cdot 10^{-3}$	$4.5 \cdot 10^{-3}$	$1.4 \cdot 10^{-3}$
23	$5.5 \cdot 10^{-5}$	$1.9 \cdot 10^{-4}$	$1.7 \cdot 10^{-4}$	$1.6 \cdot 10^{-4}$	$1.8 \cdot 10^{-4}$

⇒ **Better accuracy** if # points in the outer integral increases

⁶[B.B. Guzina and Pak, QJMAM, 2001])

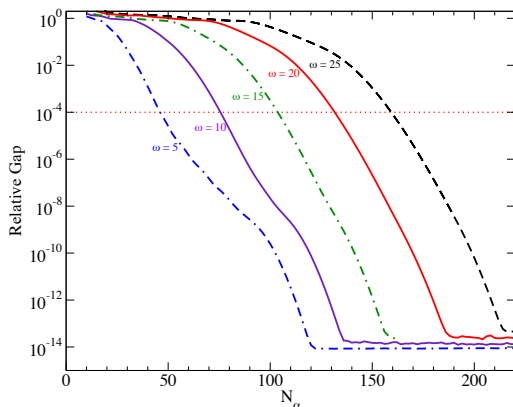
$$U_C(\mathbf{x}, \mathbf{y}) = \int_0^{\xi_{max}} \underbrace{\int_0^{2\pi} e^{i\xi(\cos \alpha(y_1 - x_1) + \sin \alpha(y_2 - x_2))} \hat{U}_C(\xi \cos \alpha, \xi \sin \alpha, y_3, x_3) d\alpha}_{I_C(\xi, \mathbf{x}, \mathbf{y})} \xi d\xi$$

Choice of N_α ? depends on Rayleigh wavenumber k_r

$$\xi_{max} = 2k_r + \frac{10}{|\min(y_3)|}; \quad \forall \xi \geq \xi_{max} \quad |I_C(\xi, \mathbf{x}, \mathbf{y})| \ll 10^{-6}$$

$$\mathbf{U}_C(\mathbf{x}, \mathbf{y}) = \int_0^{\xi_{\max}} \underbrace{\int_0^{2\pi} e^{i\xi(\cos \alpha(y_1 - x_1) + \sin \alpha(y_2 - x_2))} \hat{\mathbf{U}}_C(\xi \cos \alpha, \xi \sin \alpha, y_3, x_3) d\alpha}_{\mathbf{I}_C(\xi, \mathbf{x}, \mathbf{y})} \xi d\xi$$

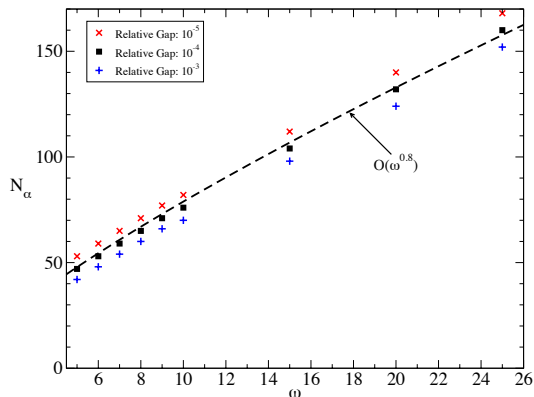
Choice of N_α ? depends on circular frequency ω



Relative Discrepancy on $\sum_j \mathbf{U}_C(\mathbf{x}_i, \mathbf{y}_j) \mathbf{t}(\mathbf{y}_j)$ with a reference solution ($N_\alpha = 250$)

$$U_C(\mathbf{x}, \mathbf{y}) = \int_0^{\xi_{\max}} \underbrace{\int_0^{2\pi} e^{i\xi(\cos \alpha(y_1 - x_1) + \sin \alpha(y_2 - x_2))} \hat{U}_C(\xi \cos \alpha, \xi \sin \alpha, y_3, x_3) d\alpha}_{I_C(\xi, \mathbf{x}, \mathbf{y})} \xi d\xi$$

Selection rule for N_α

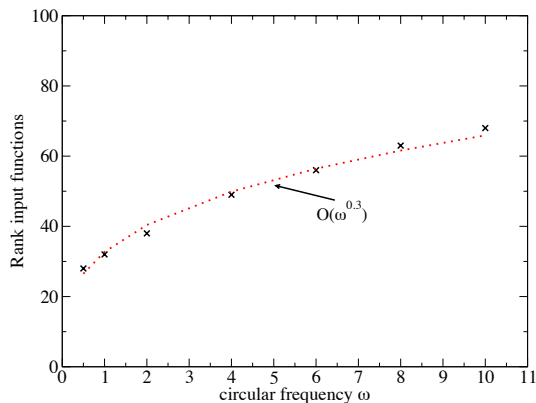


$$O(\omega) = O(N^{1/2})$$

$$\Rightarrow N_\alpha = O(N^{0.4})$$

$$u_C(\mathbf{x}, \mathbf{y}) = \int_0^{\xi_{\max}} \sum_{j=1}^{N_\alpha} w_j^\alpha e^{i\xi(\cos \alpha_j(y_1 - x_1) + \sin \alpha_j(y_2 - x_2))} \hat{u}_C(\xi \cos \alpha_j, \xi \sin \alpha_j, y_3, x_3) \xi d\xi$$

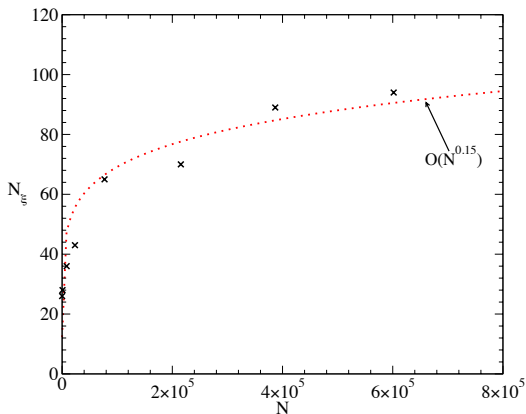
Rank of the input functions



Rank of input funct.
is $O(\omega^{0.3})$

$\Rightarrow N_\xi$ is expected to be of order $N_\xi = O(\omega^{0.3}) = O(N^{0.15})$

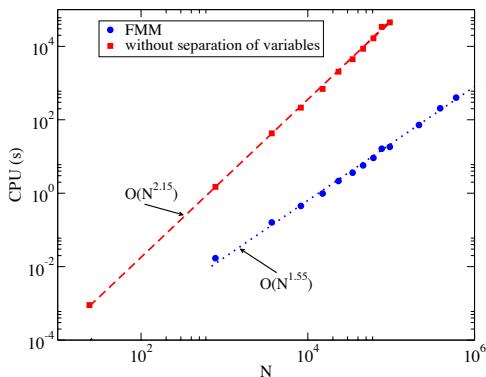
Complexity of the number of points with N



⇒ Custom-tailored algorithm leads to $N_{\xi} = O(N^{0.15})$

⇒ The anticipated trend is verified a posteriori

Theoretical complexity estimates to compute \mathbf{v}_C



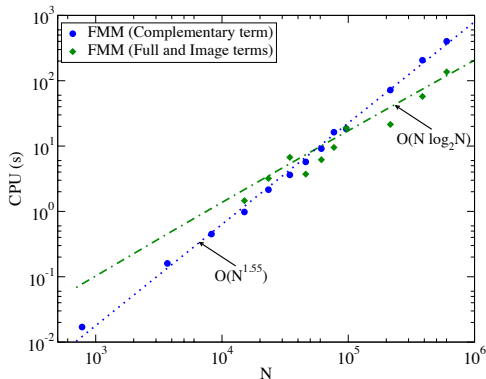
$$N_{\xi} = O(N^{0.15})$$

$$N_{\alpha} = O(N^{0.4})$$

- Without separation of variables \mathbf{x}, \mathbf{y} : $O(N_{\xi} N^2) \Rightarrow O(N^{2.15})$
- FMM: $O(N_{\alpha} N_{\xi} N) \Rightarrow O(N^{1.55})$

\Rightarrow Good **agreement** between numerical and theoretical complexity

Theoretical complexity estimate to compute $\mathbf{v} = \mathbf{v}_\infty + \bar{\mathbf{v}}_\infty + \mathbf{v}_C$

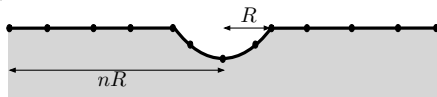


- "New" FMM to compute \mathbf{v}_C : $O(N^{1.55})$
- "Standard" FMM to compute sum of \mathbf{v}_∞ & $\bar{\mathbf{v}}_\infty$: $O(N \log N)$

\Rightarrow Similar order of magnitude to compute the various terms

Comparison of the number of DOFs: diffraction of a plane wave by a semi-spherical canyon

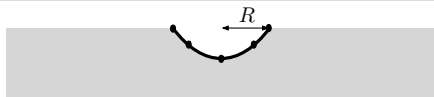
Full-space F.S.



Artificial truncation of the free-surface
 N_F

$$A_F = \pi R^2(n^2 + 1)$$

Half-space F.S.



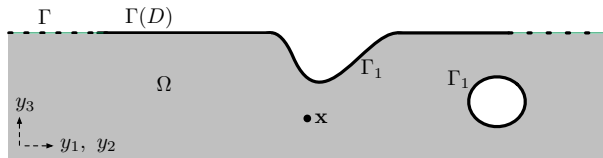
No truncation
 $N_{H.S.}$

$$A_{H.S.} = 2\pi R^2$$

- Due to geometrical considerations $N_F = \frac{n^2+1}{2} N_{H.S.}$
- If $n = 5$ (usual used value) $N_F = 13 N_{H.S.}$

- 1 Motivations and background
- 2 Formulation of the Elastic Half-space Fundamental solutions
- 3 New Fast Multipole Formulation for the Elastic Half-space
- 4 Numerical experiments and complexity study
- 5 Conclusions and future work**

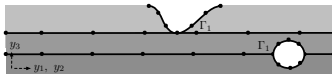
- Formulation of the **multipole expansions** for the half-space elastodynamic fundamental solutions
- Tailored algorithm to compute **Generalized Gaussian Quadrature**
- **New FMM** proposed for problems in an elastic half-space
- **Accuracy** of multipole expansion tested against published results
- **Complexity** of a matrix-vector product checked numerically



- **Implementation** of complete FMM for the elastic half-space F.S.
 - Efficient way to compute the image term?
 - Singularities when **x** and **y** are both on the free surface?
- Comparisons of **accuracy and numerical efficiency** with classical formulation based on elastic full-space F.S.
 - Can we run higher frequencies than with previous formulation?
 - Influence of truncation of the free-surface for full-space formulation?
- **Parallelization** to run even higher frequencies
- **Preconditioning**: algebraic preconditioners are only moderately efficient

- Current approach with homogeneous F.S.:
 - Boundary Element-Boundary Element coupling
 - Mesh all interfaces
 - Conforming elements because strong coupling
- Usefulness of a FMM for the **layered visco-elastic half-space** F.S. (to be formulated)
 - Mesh drastically reduced
 - Not straightforward but a similar approach should be applicable

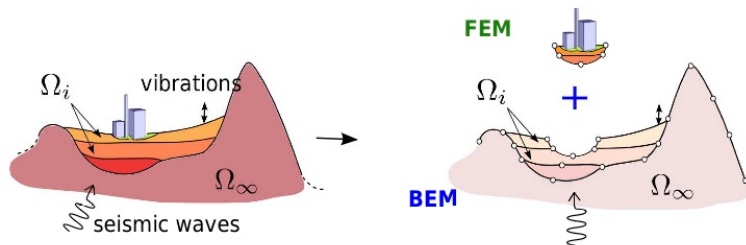
Homogeneous F.S.



Layered Half-space F.S.



Exploits complementarity between BEM and FEM



- **Finite Element Method**

- Near field
- Bounded domains with complex geometries, heterogeneities

- **Fast Multipole Accelerated Boundary Element Method**

- Far field
- Semi-infinite linear and piecewise homogeneous domains

Non-reflecting boundary conditions

- High-order Absorbing boundary conditions
- Perfectly Matched Layer
- FEM/BEM coupling

Coupling FEM/FM-BEM

- Iterative DDM-like algorithm
- BEM influence matrix not assembled

Coupling with full-space formulation: PhD thesis Eva Grasso (2012)

- Sequential approach: **easy** to implement (black box) but **slow**
 - Alternate solution in Ω_F and Ω_B
 - Update boundary conditions on the interface until convergence
- Simultaneous approach: **fast** but **involved** implementation
 - Build a global system of equations
 - Simultaneous solution of the global system

Coupling with half-space formulation

- Promising
- But involved implementation if simultaneous approach used

- Frequency-domain computations at sampling frequencies and Fourier transform
 - Depends strongly on a proper choice of parameters
- Use of the **time-domain F.S.**
 - FMM for full-space F.S. is already involved
- Other Possible approach: **Convolution Quadrature Method**
 - More stable time stepping procedure
 - Already used for visco-elastodynamics BEM accelerated by ACA (group of M. Schanz, T.U. Graz)

- Soil-structure interaction
- Seismic wave propagation
- Site-effects
- Traffic- or excavation-induced vibrations

Méthode d'équation intégrale accélérée par la méthode multipôle rapide pour les problèmes tridimensionnels de propagation d'ondes élastiques dans un demi-espace

Stéphanie Chaillat

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Séminaire LaMSID
Mardi 4 Septembre 2012

